Recursively Reducible Structures in High-Genus Riemann Theta Functions

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Technical Note / Conceptual Supplement

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1. Motivation

The numerical evaluation of the Riemann theta function is well known to suffer from the curse of dimensionality.
Given a genus (g), the computational cost grows exponentially as $(O((2N+1)^g))$, making exact evaluation practically infeasible beyond modest dimensions (typically $(g <= 10)$) in standard settings.
While various structural simplifications are known in special cases—such as block-diagonal factorization or degenerations—these are generally regarded as either trivial reductions or limiting cases, and do not provide a framework for tractable computation in genuinely high-dimensional regimes.

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2. Observation

In the course of implementing a structure-dependent evaluation algorithm based on S(2,2)-type decomposition, we observe the following phenomenon:
There exist classes of period matrices for which the effective dimensionality of the theta evaluation can be reduced recursively, leading to drastic improvements in computational tractability.
More precisely, for certain structured period matrices (Ω), the evaluation process admits a hierarchical decomposition into lower-dimensional subproblems, which can be applied repeatedly.
As a result, the overall computational behavior deviates significantly from the expected exponential scaling, and exhibits polynomial-like or near-linear growth in practical regimes, even for extremely large genus (e.g., $(g > 10^4)$).

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3. Relation to Existing Structures

This phenomenon is not entirely disconnected from known theory, but does not appear to be explicitly characterized in existing frameworks.

- In Riemann Theta Function Theory, simplifications occur under special structures such as block-diagonal forms or degenerations.
- In Integrable Systems, structured period matrices arise in finite-gap solutions and related constructions.

- In the theory of Siegel moduli spaces, special loci corresponding to algebraic or geometric constraints are well studied.

However, the observed behavior differs in the following sense:
The reduction in computational complexity occurs without requiring full block-diagonal factorization or taking a degeneration limit.

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4. Distinguishing Features

The observed structures appear to satisfy:

- They are not trivially block-diagonalizable in the standard sense
- They are not degeneration limits in the usual geometric framework
- They admit recursive reduction in effective dimensionality during evaluation
- They enable exact (non-asymptotic) computation under structural constraints

This places them in an intermediate regime between classical reducible cases and fully generic high-dimensional configurations.

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5. Terminology (Provisional)

To describe this behavior, we introduce the following provisional term:
Recursively Reducible Theta Structures
This term is intended to denote subclasses of period matrices for which theta evaluation admits recursive dimensional reduction in practice.
We emphasize that this is an observational and computational characterization, rather than a fully formalized mathematical definition at this stage.

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6. Computational Implications

Within this subclass, the computational complexity of theta evaluation appears to transition from exponential to effectively polynomial scaling.
This has several implications:

- Feasibility of exact evaluation in ultra-high-dimensional regimes
- Construction of reliable benchmark values for structured cases
- Potential use as a reference baseline for approximation methods

The associated implementation demonstrates stable and efficient evaluation for dimensions exceeding ($g = 20{,}000$) under the assumed structural conditions.

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7. Outlook and Open Questions

The observations presented here suggest several directions for further investigation:

- Formal characterization of recursively reducible structures

- Relationship to known loci in Siegel space
- Criteria for distinguishing non-trivial reducibility from hidden factorization
- Extension to approximately structured (perturbed) cases
- Applications to high-genus finite-gap solutions and related systems

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8. Scope and Limitations

This note does not claim:

- A general solution to high-dimensional theta evaluation
- A complete classification of such structures
- A proof that the observed reduction is fundamentally new in all aspects

Rather, it aims to isolate and describe a computationally significant regime that appears to be underexplored in existing literature.

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9. Associated Implementation

A practical implementation based on S(2,2) decomposition is available:
https://github.com/Moriyamax/s22-theta-acceleration
This implementation serves as an empirical foundation for the observations described above.

Summary.

We report the observation of a subclass of structured period matrices for which high-genus Riemann theta functions admit recursive dimensional reduction in evaluation, enabling exact computation in regimes previously considered intractable. This suggests the presence of an underexplored computational structure within high-dimensional theta theory.